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STUDY OF INVERTED PENDULUM USING CONVENTIONAL AND OPTIMAL CONTROLLER

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ABSTRACT

In this paper modeling of an inverted pendulum is done using fundamental law of motions (Newton's second Law of motions) for stabilization of the pendulum. The controller gain is designed through Pole Placement design (A conventional controller) and Linear Quadratic Regulator (An optimal controller) techniques and also the results for both the controller are compared. An advantage of Quadratic Control method over the pole-placement techniques is that the former provides a systematic way of computing the state feedback control gain matrix.LQR controller is designed by the selection on choosing .The proposed system extends classical inverted pendulum by incorporating two moving masses. The motion of two masses that slide along the horizontal plane is controllable .The results of computer simulation for the system with Linear Quadratic Regulator (LQR) & Pole Placement Design.

Keywords: *Inverted Pendulum, Mathematical Modeling, Pole Placement Design, Linear Quadratic Regulator (LQR), State Feedback Matrix.*

I. INTRODUCTION

In our childhood we were trying to balance a broom-stick on our index finger or the palm of our hand for fun. In that playing trick we had to constantly adjust the position of our hand to keep the object upright. An Inverted Pendulum does basically the same thing. But in case of an Inverted Pendulum the motion is restricted to one dimension only, where as in case of a broom-stick the hand is free to move in any directions. Just like the broom-stick, an Inverted Pendulum is an inherently unstable system. Force must be properly applied to keep the system intact. To achieve this, proper control theory is required. The Inverted Pendulum is a non-linear time variant open loop system. .So the standard linear techniques cannot model the non-linear dynamics of the system. This makes the system more challenging for analysis. The dynamics of the actual non-linear system is more complicated. But this non linearized system can be approximated as a linear system if the operating region is small, i.e. the variation of the angle from the normal position. Here we use two techniques, 1st is pole placement design and 2nd is Linear Quadratic Regulator

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II. MATHEMATICAL MODELING OF THE SYSTEM

Here we consider only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force **F** is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Below are the free-body diagrams of the two elements of the inverted pendulum system.

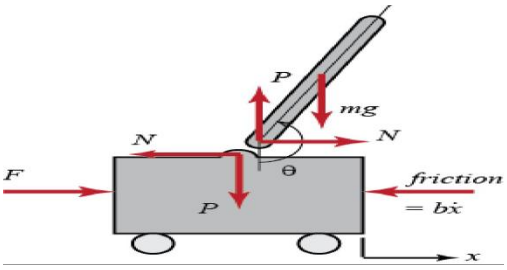

Fig. 1

Summing the forces in the free-body diagram of the cart in the horizontal direction, you get the following equation of motion.

$M\ddot{x}$ + b$\dot{x}$ + N = F (1)

Summing the forces in the free-body diagram of the pendulum in the horizontal direction, we get the following expression for the reaction force N.

$N = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$ (2)

If we substitute this equation into the first equation, we get one of the two governing equations for this system.

$(M + m)\ddot{x} + b\dot{x} + \; ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$ (3)

To get the second equation of motion for this system, sum the forces perpendicular to the pendulum. Solving the system along this axis greatly simplifies the mathematics. we should get the following equation.

$P\sin\theta + N\cos\theta - mg\sin\theta = ml\ddot{\theta} + m\ddot{x}\cos\theta$ (4)

To get rid of the P and N terms in the equation above, sum the moments about the centroid of the pendulum to get the following equation.

$-Plsin\theta \; - Nl\cos\theta = I\ddot{\theta}$ (5)

Combining these last two expressions, we get the second governing equation.

$(I + ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$ (6)

Since the analysis and control design techniques we will be employing in this example apply only to linear systems, this set of equations needs to be linearized. Specifically, we will linearize the equations about the vertically upward equilibrium position, $\theta = \pi$, and will assume that the system stays within a small neighbourhood of this equilibrium. This assumption should be reasonably valid since under control we desire that the pendulum not deviate more than 20 degrees from the vertically upward position. Let ϕ represent the deviation of the pendulum's position from equilibrium, that is, $\theta = \pi + \phi$. Again presuming a small deviation (ϕ) from equilibrium, we can use the following small angle approximations of the nonlinear functions in our system equations

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$$\cos\theta = \cos(\pi + \phi) \approx -1 \qquad (7)$$

$$\sin\theta = \sin(\pi + \phi) \approx -\phi \qquad (8)$$

$$\dot{\theta}^2 = \dot{\phi}^2 \approx 0 \qquad (9)$$

After substituting the above approximations into our nonlinear governing equations, we arrive at the two linearized equations of motion. Note u has been substituted for the input F.

$$(I + ml^2)\ddot{\phi} - mgl\phi = ml\ddot{x} \qquad (10)$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u \qquad (11)$$

2.1 Transfer Function

To obtain the transfer functions of the linearized system equations, we must first take the Laplace transform of the system equations assuming zero initial conditions. The resulting Laplace transforms are shown below.

$$(I + ml^2)\Phi(s)s^2 - mgl\Phi(s) = mlX(s)s^2 \qquad (12)$$

$$(M + m)X(s)s^2 + bX(s)s - ml\Phi(s)s^2 = U(s) \qquad (13)$$

Recall that a transfer function represents the relationship between a single input and a single output at a time. To find our first transfer function for the output $\Phi(s)$ and an input of *U(s)* we need to eliminate X(s) from the above equations. Solve the first equation for X(s).

$$X(s) = \left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s) \qquad (14)$$

Then substitute the above into the second equation.

$$(M + m)\left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s)s^2 + b\left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s)s - ml\Phi(s)s^2 = U(s) \qquad (15)$$

Rearranging, the transfer function is then the following

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 + \frac{b(I+ml^2)}{q} - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s} \qquad (16)$$

Where, $q = [(M + m)(I + ml^2) - (ml)^2]$ (17)

From the transfer function above it can be seen that there is both a pole and a zero at the origin. These can be canceled and the transfer function becomes the following.

$$P_{pend}(s) = \frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl}{q}s - \frac{bmgl}{q}} \quad \left[\frac{rad}{N}\right] \qquad (18)$$

Second, the transfer function with the cart position X(s) as the output can be derived in a similar manner to arrive at the following.

$$P_{cart}(s) = \frac{X(s)}{U(s)} = \frac{\frac{(I+ml^2)s^2 - gml}{q}}{s^4 + \frac{b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s} \quad \left[\frac{m}{N}\right] \qquad (19)$$

2.2 State Space Model

The linearized equations of motion from above can also be represented in state-space form if they are rearranged into a series of first order differential equations. Since the equations are linear, they can then be put into the standard matrix form shown below.

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\phi}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ \frac{I+ml^2}{I(M+m)+Mml^2}\\ 0\\ \frac{ml}{I(M+m)+Mml^2}\end{bmatrix}u \quad (20)$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u \quad (21)$$

The C matrix has 2 rows because both the cart's position and the pendulum's position are part of the output. Specifically, the cart's position is the first element of the output **y** and the pendulum's deviation from its equilibrium position is the second element of **y**.

III. DESIGN REQUIREMENT

Our problem is to have a closed loop system having an overshoot of 10% and settling time of 1 sec. Since the overshoot

Table 1. Parameters under considerations

Parameter	**Value**	**Unit**
Cart mass(M)	1	Kilogram
Mass of the pendulum(m)	0.2	Kilogram
Half Length of pendulum(l)	0.45	meter
Coefficient of frictional force(b)	0.1	Ns/m
Moment of inertia of pendulum(I)	0.0135	Kg/m^2
Gravitation force(g)	9.8	m/s^2

IV. OPEN-LOOP STABILITY

In this problem, represents the step command of the cart's position. The 4 states represent the position and velocity of the cart and the angle and angular velocity of the pendulum. The output contains both the position of the cart and the angle of the pendulum. We want to design a controller so that when a step reference is given to the system, the pendulum should be displaced, but eventually return to zero (i.e. vertical) and the cart should move to its new commanded position. As we all know Inverted Pendulum is inherently unstable system. The pole locations are 0, -0.0833, -4.3266, 4.3147.

We can see one of pole in right half (pole 4.3147). So system is unstable.

V. POLE PLACEMENT

- Assumptions
  - The system is completely state controllable
  - The state variables are measurable and are available for feedback.
  - Control input is unconstrained.

- Objective

The closed loop poles should lies at desired locations which are as per design requirement. Let us assume that we decide that the desired closed-loop poles are to be at s=μ_1, s=μ_2…s= μn. By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is complete state controllable. Consider a control system

$\dot{x} = Ax + Bu$

$$y = Cx + Du$$

A = n × n constant matrix

B = n × 1 constant matrix

C = 1 × n constant matrix

D = constant (scalar)

u = control signal (scalar)

y = output signal (scalar)

x = state vector (n-vector)

Where we shall choose the control signal to be

u = – Kx

This means that the control signal is determined by an instantaneous state. Such a scheme is called state feedback. The 1× n matrix K is called the state feedback gain matrix. We assume that all state variables are available for feedback. In the following analysis we assume that is unconstrained. A block diagram for this system is shown.

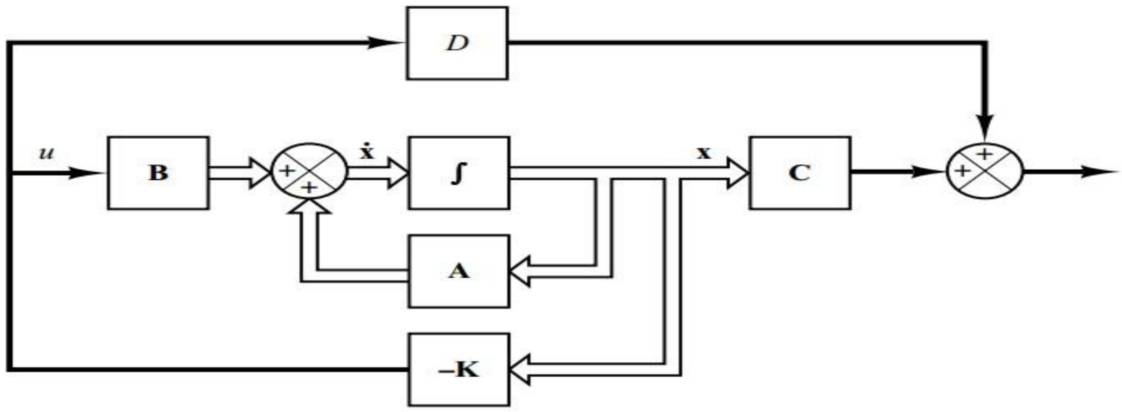

Fig.2 State space design

This closed-loop system has no input. Its objective is to maintain the zero output. Because of the disturbances that may be present, the output will deviate from zero. The nonzero output will be returned to the zero reference input because of the state feedback scheme of the system. Such a system where the reference input is always zero is called a regulator system. (Note that if the reference input to the system is always a nonzero constant, the system is also called a regulator system.)The solution of this equation is given by

$x(t) = e^{(A-BK)t}x(0)$

Where $x(0)$ is the initial state caused by external disturbances. The stability and transient response characteristics are determined by the Eigen values of matrix **[A-BK].**

If matrix x(t) =(A-BK)x(t)

K is chosen properly, the matrix **[A-BK]** can be made an asymptotically stable matrix, And for all **x(0)≠0**, it is possible to make x(t)approach **0** as **t** approaches infinity. The eigen values of matrix **[A-BK]** are called the regulator poles. If these regulator poles are placed in the left-half s plane, then **x(t)** approaches to infinity. The problem of placing the regulator poles (closed-loop poles) at the desired location is called a Pole-placement problem.

Determination of Matrix K Using Ackermann's Formula:

There is a well-known formula, known as Ackermann's formula, for the determination of the state feedback gain matrix K. We shall present this formula in what follows.

Consider the system

$$\dot{x} = Ax + Bu$$

Where we use the state feedback control **u = –Kx.** We assume that the system is completely state controllable.

We also assume that the desired closed-loop poles are at s = μ_1, s=μ_2…S= μ_n

Use of the state feedback control

u=−Kx modifies the system equation to

$$\dot{x} = (A - BK)x$$

$$\tilde{A} = A - BK$$

The desired characteristic equation is

$$|sI - A + BK| = |sI - \tilde{A}| = (s + \mu_1)(s + \mu_2) \dots \dots (s + \mu_n)$$

From Cayley –Hamilton theorem we get the value of gain matrix.

$$K = [0 \quad 0 \cdots 0 \quad 1][B \vdots AB \vdots \cdots A^{n-1}B]^{-1}\phi(A)$$

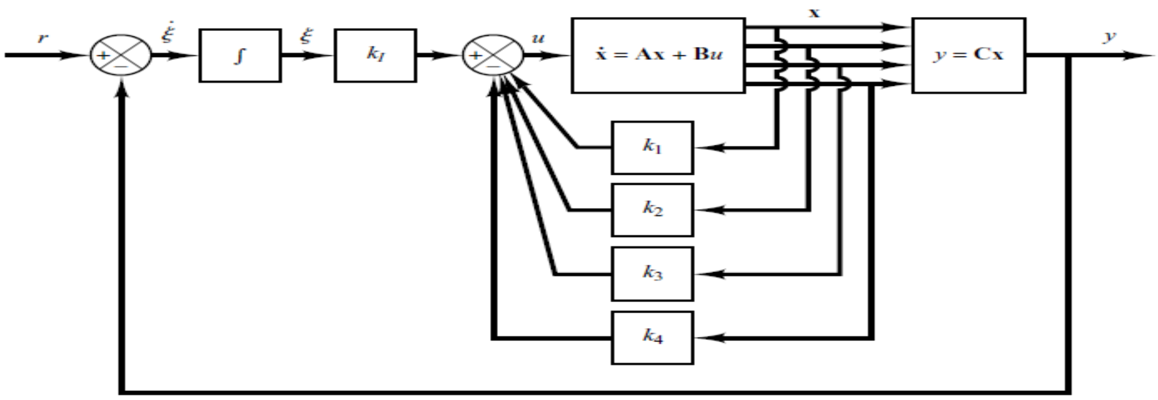

Fig.3 Inverted-pendulum control system.

Analytical Calculations of Inverted Pendulum by using Pole Placement design.

As per design requirement maximum overshoot 10 % and settling time 1sec. Our pole is calculated below

$$MP = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.1$$

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Solving above equation we get value of $\zeta = 0.591$.

Settling time = $\frac{4}{\zeta \omega_n}$

Solving above equation we get value of $\omega_n = 6.768$.

From the above value we get value of poles

-4+ 5.459j, -4-5.459j, -10, -20

VI. LINEAR QUADRATIC REGULATOR

Deriving the state X of a linear system.

$\dot{X} = AX + BU$

To the origin by minimizing the following quadratic performance Index (cost function)

$$J = \frac{1}{2}(X_f{}^T S_f X_f) + \int_{t_0}^{t_f} (X^T QX + U^T RU)dt$$

$S_f, Q \geq 0(psdf), R > 0(pdf)$

Necessary condition of Optimality

- Terminal Penalty $\phi(X_f) = \frac{1}{2}(X_f{}^T S_f X_f)$
- Hamiltonian $H = \frac{1}{2}(X^T QX + U^T RU) + \lambda^T(AX + BU)$
- State Equation $\dot{X} = AX + BU$
- Costate Equation $\dot{\lambda} = -\frac{\partial H}{\partial X} = -(QX + A^T\lambda)$
- Optimal Control Equation $\left(\frac{\partial H}{\partial U}\right) = 0 \Rightarrow U = -R^{-1}B^T\lambda$
- Boundary Condition $\lambda_f = \frac{\partial \phi}{\partial X_f} = S_f X_f$

Riccati Equation

$PA + A^T P - PBR^{-1}B^T P + Q = 0$

Solution Procedure

- Use the boundary condition $P(t_f) = S_f$ and integrate the Riccati Equation backwards from t_f *to* t_0.
- Store the solution history for Riccati matrix.
- Compute the optimal control online

$U = -(R^{-1}B^T P)X = -KX$

Infinite Time Regulator Problem

Theorem(By Kalman)

As $t_f \to \infty$, for constant Q and R matrices, $\dot{P} \to 0 \; \forall \; t$

Algebraic Riccati Equation(ARE)

$PA + A^T P - PBR^{-1}B^T P + Q = 0$

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As here we can see that on changing the weight cost matrix Q and R. Settling time also changes. Decreasing the value Q causing system instability, so proper choice of Q and R is needed.

Q= [1 0 0 0; 0 0 0 0;0 0 1 0;0 0 0 0], R= [.001]

We get the optimum result.

VII. SIMULATION & RESULTS

By using MATLAB we got the outputs

Simulation result using pole placement design:

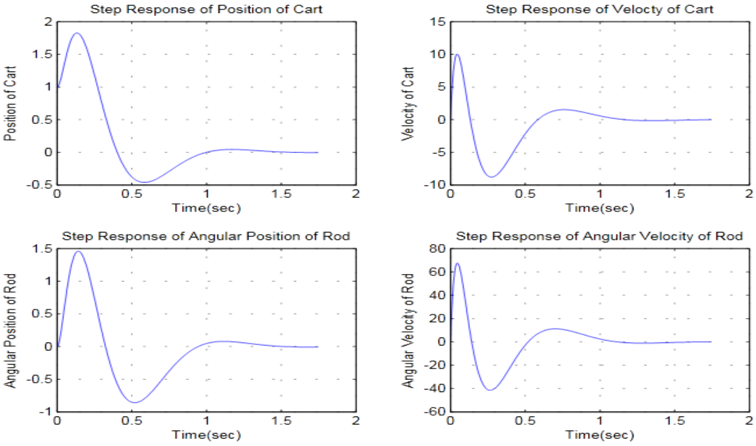

Fig.4 Response of Inverted Pendulum using Pole Placement Design

Simulation result using LQR:

Q = [1 0 0 0; 0 0 0 0; 0 0 1 0; 0 0 0 0]

R = [0.001]

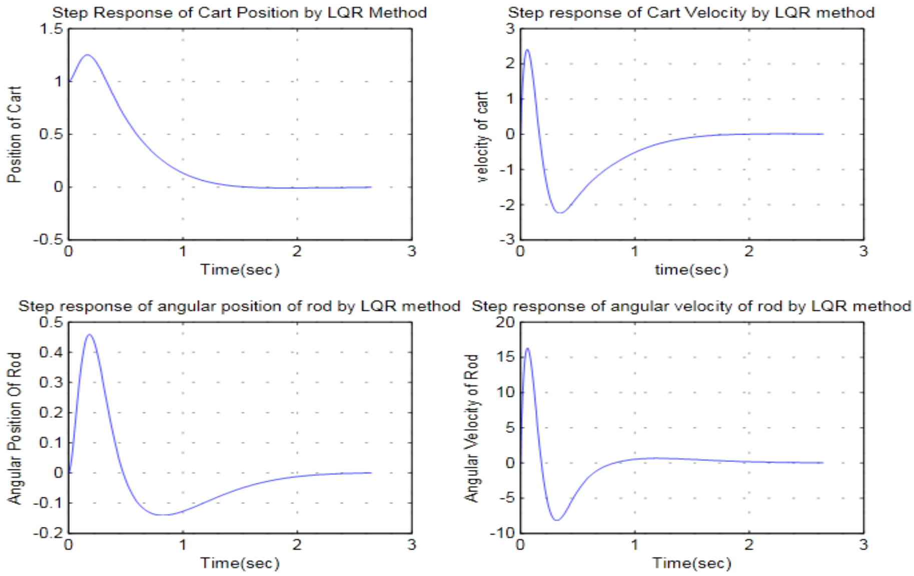

Fig. 5. Response of Inverted Pendulum using LQR

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VIII. CONCLUSION

Modeling of inverted pendulum shows that system is unstable. Results of applying state feedback controllers show that the system can be stabilized. LQR controller method is cumbersome because of selection of constants of controller. Constant of the controllers can be tuned by some soft computing techniques for better result.

Comparing the step response characteristics of this system with those of pole placement we have observed that the response of present system is less oscillatory and exhibits less maximum overshoot .The system designed by quadratic optimal regulator approach is- less oscillatory and well damped

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